8. Receiver Operating Characteristic (ROC) Curve

ECE 830, Spring 2014

Recap of the signal detection problem

The binary hypothesis test

$$H_0: \ X = W$$
$$H_1: \ X = S + W$$

where $W \sim N(0, \sigma^2 I_{n\times n})$ and $S = [s_1, s_2, \ldots, s_n]^T$ is the known signal.

$$P_0(X) = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} exp\left(-\frac{1}{2\sigma^2} X^T X\right)$$
$$P_1(X) = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} exp\left[-\frac{1}{2\sigma^2}(X-S)^T(X-S)\right]$$

Apply the likelihood ratio test (LRT):

$$\log \Lambda(x) = \log \frac{P_1(X)}{P_0(X)} = -\frac{1}{2\sigma^2}[-2X^T S + S^T S] \underset{H_0}{\overset{H_1}{\gtrless}} \gamma'$$

After simplification, we have

$$X^T S \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \gamma' + \frac{S^T S}{2} = \gamma$$

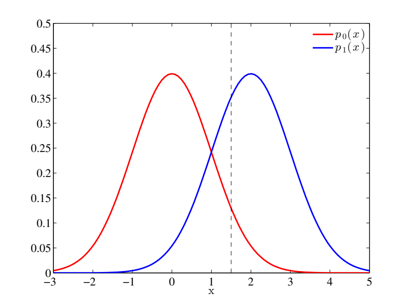
The test statistic $X^T S$ is usually called a “matched filter”. The LR detector “filters” data by projecting them onto the signal subspace.

Quality of the classifier

Question: How can we assess the quality of a detector?

Example:

$$H_0 : \; X \sim N(0,1)$$
$$H_1 : \; X \sim N(2,1)$$



P_{FA} and P_D characterize the performance of the detector. As γ increases, P_{FA} decreases (good) and P_D decreases (bad).

Receiver Operating Characteristic (ROC) Curve

What is an ROC curve?

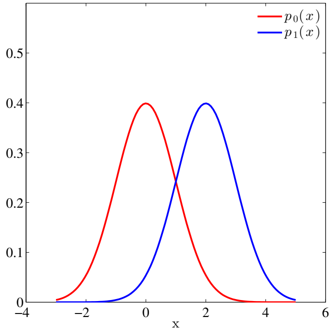
An ROC curve is a plot that illustrates the performance of a detector (binary classifier) by plotting its P_D vs. P_{FA} at various threshold settings.

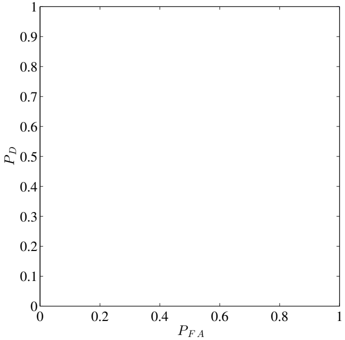
First use:

In World War II. The ROC curve was first developed by electrical engineers and radar engineers during for detecting aircrafts from radar signals after the attack on the Pearl Harbor.

Computation of the ROC curve

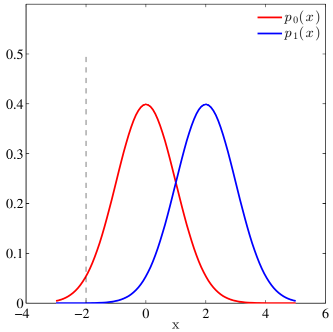
Vary the threshold level γ and compute P_{FA} and P_D.

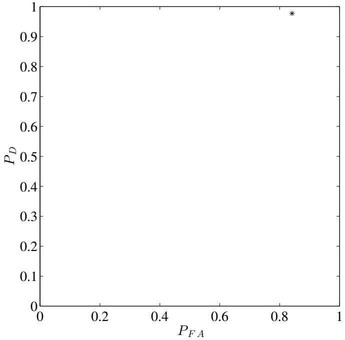




Computation of the ROC curve

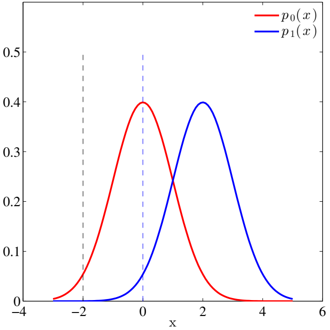
$$(P_{FA}, P_D) = (0.84, 0.98)$$

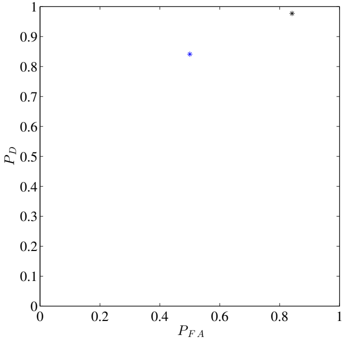




Computation of the ROC curve

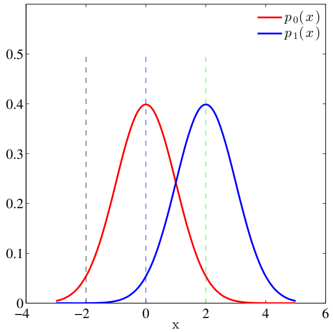
$$(P_{FA}, P_D) = (0.50, 0.84)$$

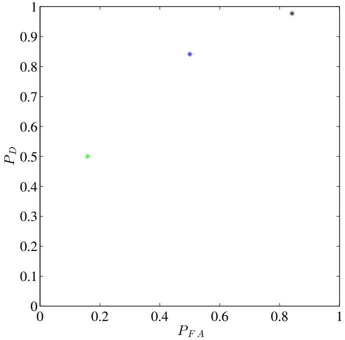




Computation of the ROC curve

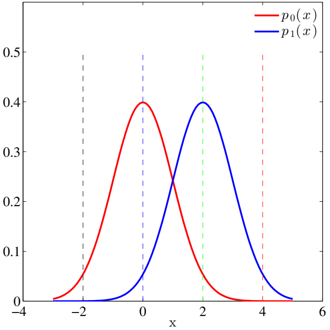
$$(P_{FA}, P_D) = (0.16, 0.50)$$

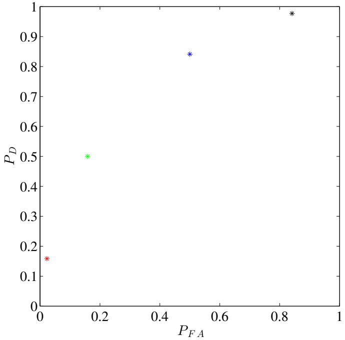




Computation of the ROC curve

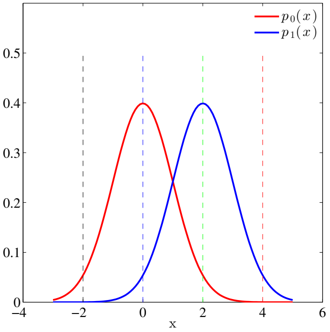
$$(P_{FA}, P_D) = (0.23, 0.16)$$

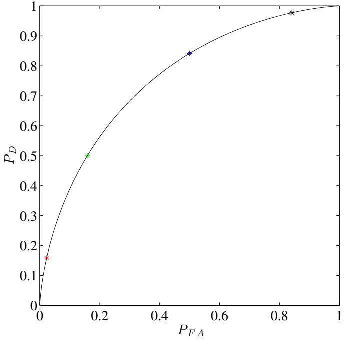




Computation of the ROC curve

Complete the curve.





The ROC curve

- Starts from $(0,0)$ and ends at $(1,1)$ (unless $p_i(\pm\infty) > 0$).
- The diagonal line from $(0,0)$ to $(1,1)$ corresponds to random guesses.
- Depends on signal strength, noise strength, noise type, etc.

Example: ROC and SNR

$$
\begin{aligned}
H_0 &: \; X \sim N(0, \sigma^2 I) \\
H_1 &: \; X \sim N(S, \sigma^2 I)
\end{aligned}
$$

The likelihood ratio test gives

$$X^T S \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

Example: ROC and SNR

$X^T S$ is also Gaussian distributed. Recall if $X \sim N(\mu, \Sigma)$, then $Y = AX \sim N(A\mu, A\Sigma A^T)$. So we can get

$$H_0 : \; X^T S \sim$$
$$H_1 : \; X^T S \sim$$

We can also compute

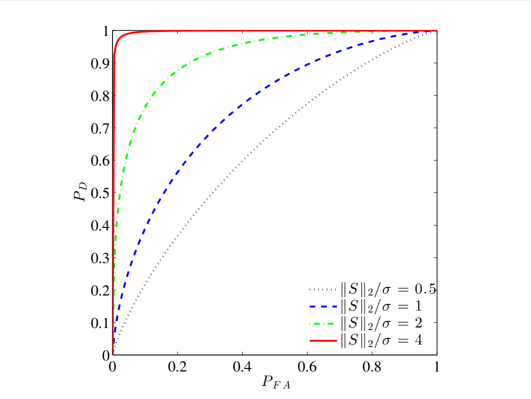
$$P_{FA} =$$

$$P_D =$$

Example: ROC and SNR

Since Q function is invertible, we can get .
Therefore,

where is the square root of Signal-to-Noise Ratio($\sqrt{SNR}$).



The AWGN Assumption

AWGN is gaussian distributed as

$$W \sim N(0, \sigma^2 I)$$

Is real-world noise really additive, white and Gaussian? Noise in many applications (e.g. communication and radar) arise from several independent sources, all adding together at sensors and combining additively to the measurement.

Central Limit Theorem

If $x_1, \ldots, x_n$ are independent random variables with means μ_i and variances $\sigma_i^2 < \infty$,then $Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{x_i - \mu_i}{\sigma_i} \to N(0,1)$ in distribution as $n \to \infty$.

Thus, it is quite reasonable to model noise as additive and Gaussian in many applications. However, whiteness is not always a good assumption.

Colored Gaussian Noise

Example: Correlated noise

$W = S_1 + S_2 + \cdots + S_k$, where $S_1, S_2, \ldots S_k$ are interferencing signals that are not of interest, and each of them is structured/correlated in time.

$W \sim N(0, \Sigma)$ is called correlated or "colored" noise, where Σ is a structured covariance matrix.
Consider the binary hypothesis test in this case.

$$H_0: \; X = S_0 + W$$
$$H_1: \; X = S_1 + W$$

where $W \sim N(0, \Sigma)$ and S_0 and S_1 are know signal. So we can rewrite the hypothesis as

$$H_0: \; X \sim N(S_0, \Sigma)$$
$$H_1: \; X \sim N(S_1, \Sigma)$$

The probability density of each hypothesis is

$$P_i(X) = \frac{1}{(2\pi)^{\frac{2}{n}}(\Sigma)^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(X - S_i)^T\Sigma^{-1}(X - S_i)\right], i = 0, 1$$

The log likelihood ratio is

$$\begin{aligned}\log\left(\frac{P_1(X)}{P_2(X)}\right) &= -\frac{1}{2}\left[(X - S_1)^T\Sigma(X - S_1) - (X - S_0)^T\Sigma^{-1}(X - S_0)\right] \\ &= X^T\Sigma^{-1}(S_1 - S_0) - \frac{1}{2}S_1^T\Sigma^{-1}S_1 + \frac{1}{2}S_0^T\Sigma^{-1}S_0 \\ &\underset{H_0}{\overset{H_1}{\gtrless}} \gamma'\end{aligned}$$

Equivalently,

$$(S_1 - S_0)^T \Sigma^{-1} X \underset{H_0}{\overset{H_1}{\gtrless}} \gamma' + \frac{S_1^T \Sigma^{-1} S_1}{2} - \frac{S_0^T \Sigma^{-1} S_0}{2} = \gamma$$

Let $t(X) = (S_1 - S_0)^T \Sigma^{-1} X$, we can get

$$\begin{aligned} H_0 &: \; t \sim N((S_1 - S_0)^T \Sigma^{-1} S_0, (S_1 - S_0)^T \Sigma^{-1} (S_1 - S_0)) \\ H_1 &: \; t \sim N((S_1 - S_0)^T \Sigma^{-1} S_1, (S_1 - S_0)^T \Sigma^{-1} (S_1 - S_0)) \end{aligned}$$

The probability of false alarm is

$$P_{FA} = Q\left(\frac{\gamma - (S_1 - S_0)^T \Sigma^{-1} S_0}{\left[(S_1 - S_0)^T \Sigma^{-1} (S_1 - S_0) \right]^{\frac{1}{2}}} \right)$$

In this case it is natural to define

$$SNR = (S_1 - S_0)^T \Sigma^{-1} (S_1 - S_0)$$

Example: ROC with colored Gaussian noise

$$S_1 = [\frac{1}{2}, \frac{1}{2}], S_0 = [-\frac{1}{2}, -\frac{1}{2}],$$
$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \Sigma^{-1} = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}.$$

The test statistics is

Example: ROC with colored Gaussian noise

The testing problem is equivalent to

The probabilities of false alarm and detection are

Example: ROC with colored Gaussian noise

The ROC curve for different ρ