11. The Generalized Likelihood Ratio

ECE 830, Spring 2014

The **generalized likelihood ratio test (GLRT)** is a general procedure for composite testing problems. The basic idea is to compare the best model in class H_1 to the best in H_0, which is formalized as follows. We have two composite hypotheses of the form:

$$H_i : X \sim p_i(x|\theta_i), \theta_i \in \Theta_i \,, i = 0, 1.$$

The parametric densities p_0 and p_1 need not have the same form. The GLRT based on an observation x of X is

$$\widehat{\Lambda}(x) = \frac{\max\limits_{\theta_1 \in \Theta_1} p_1(x|\theta_1)}{\max\limits_{\theta_0 \in \Theta_0} p_0(x|\theta_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma,$$

or equivalently

$$\log \widehat{\Lambda}(x) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma.$$

Example: Signal Detection

Consider two hypotheses

$$\begin{aligned} H_0 &: X \sim \mathcal{N}(0, \sigma^2 I_n) \\ H_1 &: X \sim \mathcal{N}(H\theta, \sigma^2 I_n) \end{aligned}$$

where $\sigma^2 > 0$ is known, H is a known $n \times k$ matrix, and $\theta \in \mathbb{R}^k$ is unknown. The mean vector $H\theta$ is a model for a signal that lies in the k-dimensional subspace spanned by the columns of H (e.g., a narrowband subspace, polynomial subspace, etc.). In other words, the signal has the representation

$$s = \sum_{i=1}^{k} \theta_i h_i, H = [h_1, \ldots, h_k].$$

The null hypothesis is that no signal is present (noise only).

Example: (cont.)

Log LR

$$\begin{aligned}\log \Lambda(x) &= -\frac{1}{2\sigma^2}(x - H\theta)^\top (x - H\theta) + \frac{1}{2\sigma^2} x^\top x \\ &= \frac{1}{\sigma^2}(\theta^\top H^\top x - \frac{1}{2}\theta^\top H^\top H\theta).\end{aligned}$$

Since θ is unknown we can't go further, instead we find θ that makes x most likely:

$$\begin{aligned}\widehat{\theta} &= \arg\max_\theta p(x|H_1, \theta) \\ &= \arg\max_\theta \frac{1}{(2\pi\sigma^2)^{\frac{k}{2}}} e^{-\frac{1}{2\sigma^2}(x - H\theta)^\top (x - H\theta)} \\ &= \arg\max_\theta -\frac{1}{2\sigma^2}(x - H\theta)^\top (x - H\theta) \\ &= \arg\min_\theta (x - H\theta)^\top (x - H\theta) \\ &= \arg\min_\theta (x^\top x - 2\theta^\top H^\top x + \theta^\top H^\top H\theta)\end{aligned}$$

Example: (cont.)

Taking the derivative with respect to θ

$$\begin{aligned}\frac{\partial}{\partial\theta}(x^\top x - 2\theta^\top H^\top x + \theta^\top H^\top H\theta) = 0 \\ \Rightarrow 0 - 2H^\top x + 2H^\top H\theta = 0 \\ \Rightarrow \widehat{\theta} = (H^\top H)^{-1}H^\top x\end{aligned}$$

Now we plug $\widehat{\theta}$ into the GLRT: $\theta \leftarrow \widehat{\theta}$

$$\begin{aligned}\log\widehat{\Lambda}(x) :=& \frac{1}{\sigma^2}\Big[x^\top H(H^\top H)^{-1}H^\top x \\ &- \frac{1}{2}x^\top H(H^\top H)^{-1}H^\top H(H^\top H)^{-1}H^\top x\Big] \\ =& \frac{1}{2\sigma^2}x^\top H(H^\top H)^{-1}H^\top x\end{aligned}$$

Example: (cont.)

Recall that the projection matrix onto the subspace is defined as $P_H := H(H^\top H)^{-1}H^\top$

$$\log \widehat{\Lambda}(x) = \frac{1}{2\sigma^2} x^\top P_H x = \frac{1}{2\sigma^2} \|P_H x\|_2^2.$$

Observe that this is simply an energy detector in H: we are taking the projection of x onto H and measuring the energy. The expected value of this energy under H_0 (noise only) is

$$\mathbb{E}_{H_0}\left[\|P_H X\|_2^2\right] = k\sigma^2,$$

since a fraction k/n of the total noise energy $n\sigma^2$ falls into this subspace.

The performance of the subspace energy detector can be quantified as follows. We choose a γ for the desired P_{FA}:

$$\frac{1}{\sigma^2} x^\top P_H x \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

What is the distribution of $x^\top P_H x$ under H_0? First use the decomposition

$$P_H = UU^\top$$

where $U \in \mathbb{R}^{n \times k}$ with orthonormal columns spanning columns of H, and let $y := U^\top x$. Then

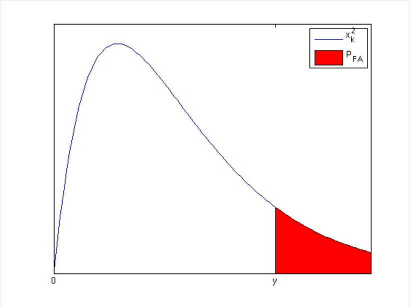
$$\begin{aligned}
\frac{1}{\sigma^2} x^\top P_H x &= \\
y &\sim \\
y_i/\sigma &\overset{iid}{\sim} \qquad\qquad\qquad , i = 1, \dots, k \\
\Rightarrow \frac{y^\top y}{\sigma^2} &\sim
\end{aligned}$$

GLRT and P_{FA}

Example: (cont.)

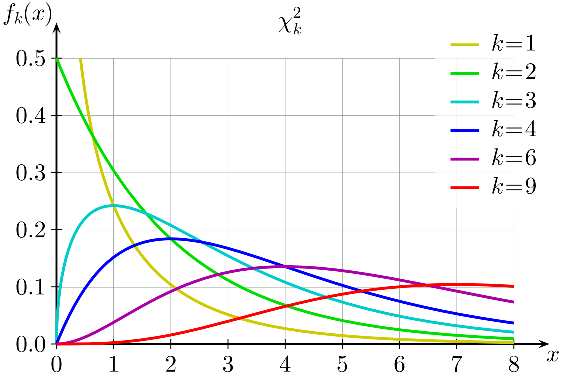
Under H_0,

$$\frac{1}{\sigma^2} x^\top P_H x \sim \chi_k^2 \qquad \Longrightarrow \qquad P_{FA} = \mathbb{P}(\chi_k^2 > \gamma)$$



The P_{FA} of a χ_k^2 distribution.

χ_k^2 Distributions



χ_k^2 distributions, for $k > 2$ they all take on the same general form. (Wikipedia)

To calculate the tails on χ_k^2 distributions you can look it up in the back of a good book or use Matlab (chi2cdf(x,k), chi2inv(γ,k), chi2cdf(x,k)). Remember the mean of a χ_k^2 distribution is k, so we want to choose a γ bigger than k to produce a small P_{FA}.

Wilks' Theorem

Wilk's Theorem (1938)

Consider a composite hypothesis testing problem

$$
\begin{aligned}
H_0 \quad &: \quad X_1, X_2, ..., X_n \overset{iid}{\sim} p(x|\theta_0), \\
&\quad \text{where } \theta_{0,1}, \ldots, \theta_{0,\ell} \in \mathbb{R} \text{ are free parameters and} \\
&\quad \theta_{0,\ell+1} = a_{\ell+1}, \ldots, \theta_k = a_k \text{ are fixed at the values} \\
&\quad a_{\ell+1}, \ldots, a_k \\
H_1 \quad &: \quad X_1, X_2, ..., X_n \overset{iid}{\sim} p(x|\theta_1), \theta_1 \in \mathbb{R}^k \text{are all free parameters}
\end{aligned}
$$

and the parametric density has the same form in each hypothesis.

In this case family of models in H_0 is a subset of those in H_1, and we say that the hypotheses are **nested**. (This is a key condition that must hold for this theorem.)

Wilk's Thm (cont.)

If the 1^{st} and 2^{nd} order derivatives of $p(x|\theta_i)$ with respect to θ_i exist and if $\mathbb{E}\left[\frac{\partial \log p(x|\theta_i)}{\partial \theta_i}\right] = 0$ (which guarantees that the MLE $\widehat{\theta}_i \to \theta_i$ as $n \to \infty$), then the generalized likelihood ratio statistic, based on an observation $X = (X_1, \ldots, X_n)$,

$$\widehat{\Lambda}_n(X) = \frac{\max\limits_{\theta_1} p(x|\theta_1)}{\max\limits_{\theta_0} p(x|\theta_0)} \tag{1}$$

has the following asymptotic distribution under H_0:

$$2 \log \widehat{\Lambda}(x) \overset{n \to \infty}{\sim} \chi^2_{k-\ell} \qquad \textit{i.e.,} \qquad 2 \log \widehat{\Lambda}(x) \xrightarrow{D} \chi^2_{k-\ell}$$

Proof: (Sketch) under the conditions of the theorem, the log GLRT tends to the log GLRT in a Gaussian setting according to the Central Limit Theorem (CLT).

Example: Nested Condition

$$H_0 : x_i \overset{iid}{\sim} \mathcal{N}(0,1)$$
$$H_1 : x_i \overset{iid}{\sim} \mathcal{N}(0,\sigma^2), i = 1, 2, \ldots, n, \sigma^2 > 0 \text{ unknown}$$

<u>log LR:</u>

<u>MLE of σ^2:</u>

$$\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

<u>log GLR under H_0:</u>