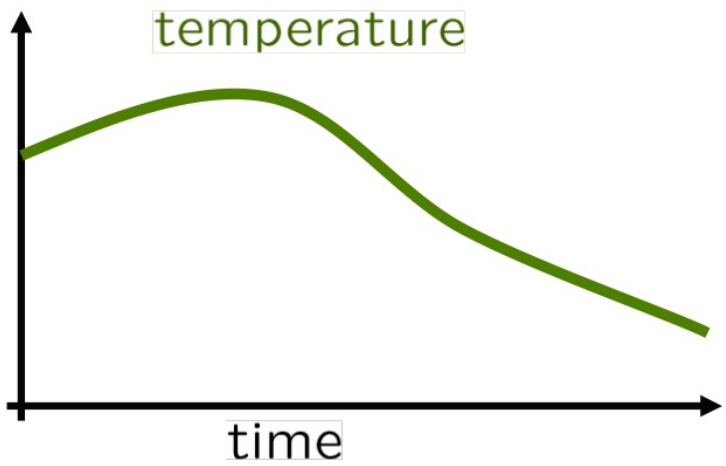
24. Density Propagation

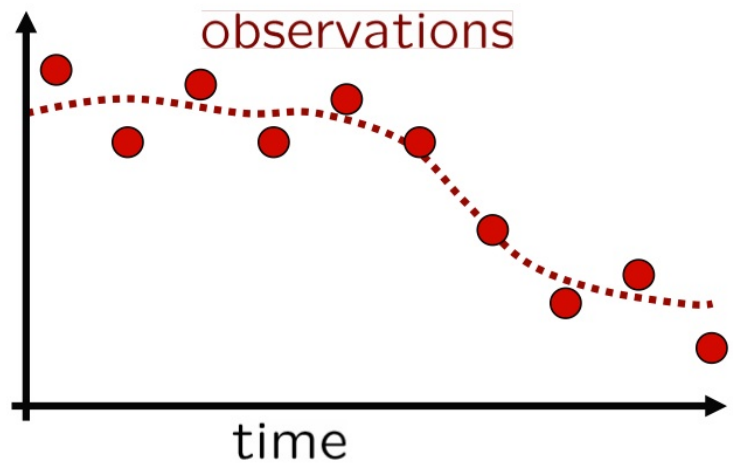
ECE 830, Spring 2014

Dynamic Filtering

In many applications we want to track a time-varying (dynamic) phenomenon.

Example: Tracking temperature or humidity in a museum room with an inaccurate device





Key: Temperature changes slowly with time so we should be able to average across time to obtain better estimates. How to do this?
Model dynamics of temperature changes and noise/uncertainties in measurement.

Dynamical State Equation (Prior)

Let x_1, x_2, ... denotes quantity ("state") of interest. The state is changing over time and we will model this variation stochastically as follows. The state at time n depends causally on the past.

Let

$$p(x_n|x_{n-1}, x_{n-2}, \ldots, x_1)$$

denote the conditional distribution of the state at time n given all the past states. This distribution is a n-variate function, and as n grows it becomes more and more complex (to specifiy, to compute, etc).

A reasonable simplifying assumption is to assume that the probability distribution of the state at time n depends only on value of the state at time $n-1$.

Defintion: Markovian assumption

$$p(x_n|x_{n-1}, \ldots, x_1) = p(x_n|x_{n-1}) \ .$$

Note that $p(x_n|x_{n-1})$ is bivariate and therefore much simpler than the general causal model.

To define the state process we must to specify

1. $p(x_1)$, the "initial state" distribution
2. $p(x_n|x_{n-1})$, $n = 2, 3, \ldots$, the state transition probability density functions

This is illustrated in the following example.

Example: Santa Tracker

On December 25th, legend has it that Santa Claus makes his way around the globe, delivering toys to all the good girls and boys. Tracking Santa's delivery trip has attracted considerable interest by the signal processing research community in recent years, see `http://www.noradsanta.org/`. Here is a simple approach to the problem.

$$\begin{aligned} x(t) &= \text{Santa's position at time } t \text{ on Christmas Eve} \\ \frac{\partial x(t)}{\partial t} &= v(t), \text{ velocity} \\ \frac{\partial v(t)}{\partial t} &= u(t), \text{ acceleration} \end{aligned}$$

We can sample Santa's position once every second, producing a sequence of position values $x_1, x_2, \ldots$. His velocity is also represented by a discrete-time process $v_1, v_2, \ldots$.

Example: (cont.)

We use the following model for Santa's dynamics:

$$\begin{bmatrix} x_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix} u_n,$$

$$u_n \sim \mathcal{N}(0,1), \quad \Delta \text{ small}$$

Also, Santa's initial position is the North Pole, denoted by x_0. So we take $p(x_1) = \delta(x_1 - x_0 - \Delta v_0)$. In words, Santa's position at time x_{n+1} is equal to his position at time n plus a small step proportional to his velocity. His velocity is modeled as a Gaussian white noise process, representing the fact that he randomly speeds up and slows down as he makes his stops around the world.

Observation Model (Likelihood)

Usually we cannot observe x_n directly. Instead we observe $z_1, z_2, \ldots,$ which are noisy and/or indirect measurements related to the states.

Example: Observation processes

$$z_n = x_n + w_n \ , \ w_n \sim \mathcal{N}(0, \sigma^2) \ , \text{ simple signal+noise model}$$

$$z_n = A \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} + w_n \ , \text{ where } A \text{ is a } 2 \times 2 \text{ matrix (e.g. blur)}$$

$$z_n = f(x_n) + w_n \ , \ f \text{ is a non-linear function}$$

Let $p(z_n|x_n)$ denote the likelihood of x_n based on observation z_n. We can combine the likelihoods and the priors $p(x_n|x_{n-1})$ to compute the posterior distribution of $x = (x_1, \ldots, x_n)$ given $z = (z_1, \ldots, z_n)$

$$p(x|z) \;\propto\; p(z|x)p(x) \;=\; \prod_{i=1}^{n} p(z_n|x_n)p(x_n|x_{n-1}) \;.$$

The posterior can be computed efficiently in an incremental fasion by exploiting Markovian structure of state transitions (prior). This incremental procedure is called *Density Propagation*.

Density Propagation

Density Propagation is an incremental procedure for efficiently computing $p(x_n|z_1, \ldots, z_n)$. First let's establish some notation:

Prior: (S is for State)

$$S_n(x_n|x_{n-1}) := p(x_n|x_{n-1}), \quad P_1(x_1) = p(x_1)$$

Likelihood: (L is for Likelihood)

$$L_n(z_n|x_n) := p(z_n|x_n)$$

Posterior: (F is for Filter)

$$F_n(x_n) := p(x_n|z_1, \ldots, z_n)$$

Prediction: (P is for Prediction or Prior)

$$P_n(x_n) := p(x_n|z_1, \ldots, z_{n-1})$$

$P_n(x_n)$ is the prediction of the value of x_n using only observations up to time $n-1$, and this will play a key role in the Density Propagation algorithm.

Density Propogation Algorithm

$n = 1$: predict x_1:

$$x_1 \sim p_1(x_1)$$

observe z_1 and compute posterior:

$$F_1(x_1) = p(x_1|z_1) = \frac{p(z_1|x_1)p(x_1)}{p(z_1)} \propto L_1(z_1|x_1)p_1(x_1)$$

$n = 2$: predict x_2:

$$\begin{aligned}
p(x_1, x_2|z_1) &= \frac{p(x_1, x_2, z_1)}{p(z_1)} \\
&= \frac{p(x_2|x_1, z_1)p(x_1|z_1)p(z_1)}{p(z_1)} \\
&= p(x_2|x_1)F_1(x_1) \\
&= S_2(x_2|x_1)F_1(x_1) \\
p(x_2|z_1) &= \int S_2(x_2|x_1)F_1(x_1)\mathrm{d}x \\
&=: P_2(x_2)
\end{aligned}$$

$n = 2$ **(cont.):** observe z_2 and compute posterior:

$$\begin{aligned} F_2(x_2) &= p(x_2|z_1, z_2) \\ &= \frac{p(x_2, z_1, z_2)}{p(z_1, z_2)} \\ &= \frac{p(z_2|x_2)p(x_2|z_1)p(z_1)}{p(z_1, z_2)} \\ &\propto L_2(z_2|x_2)P_2(x_2) \end{aligned}$$

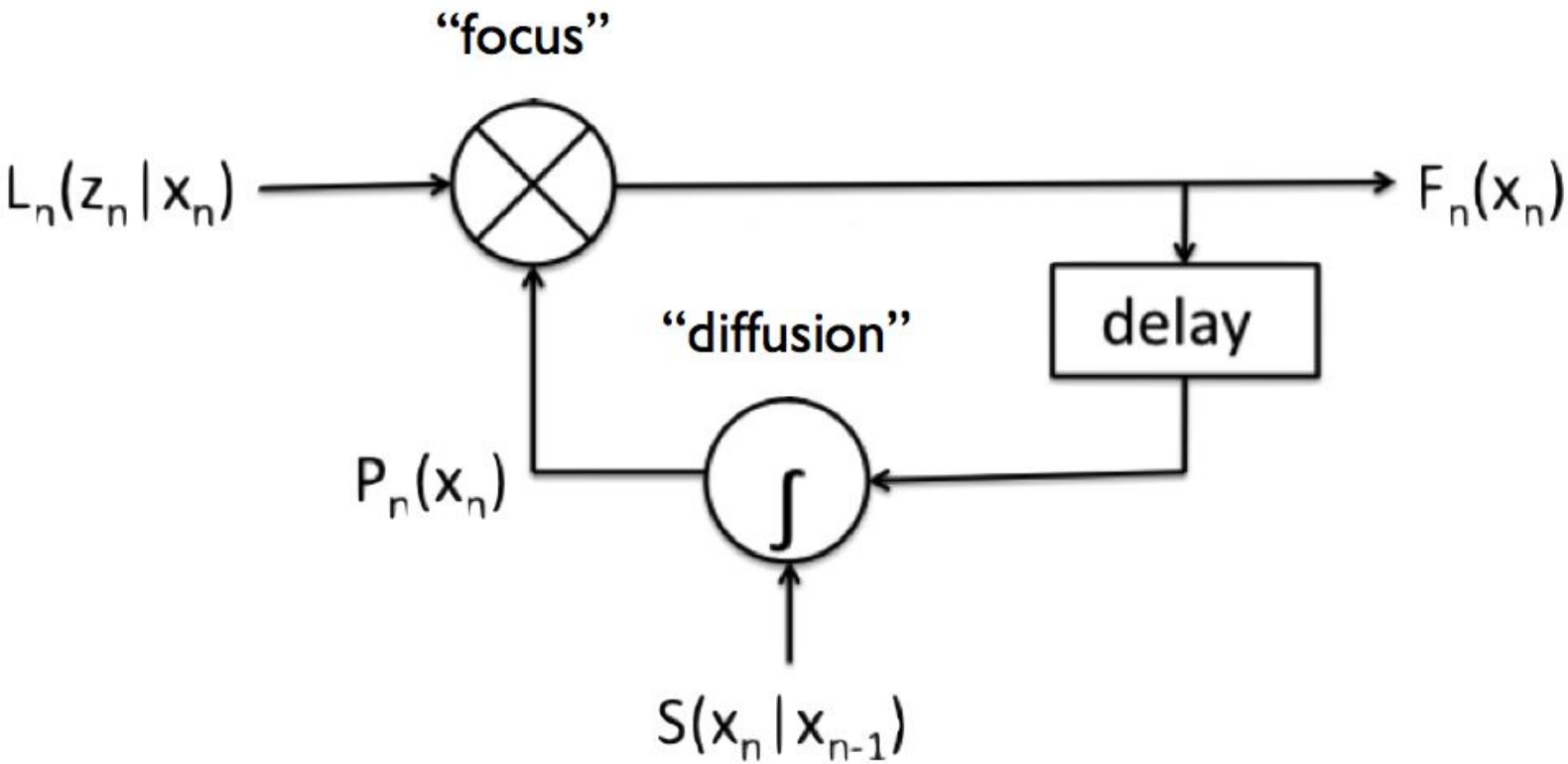
at time step n: predict x_n:

$$\begin{aligned} P_n(x_n) &= p(x_n|z_1,\ldots,z_{n-1}) \\ &= \int S_n(x_n|x_{n-1})F_{n-1}(x_{n-1})\mathrm{d}x_{n-1} \end{aligned}$$

observe z_n and compute posterior:

$$\begin{aligned} F_n(x_n) &= p(x_n|z_1,\ldots,z_n) \\ &\propto L_n(z_n|x_n)P_n(x_n) \end{aligned}$$

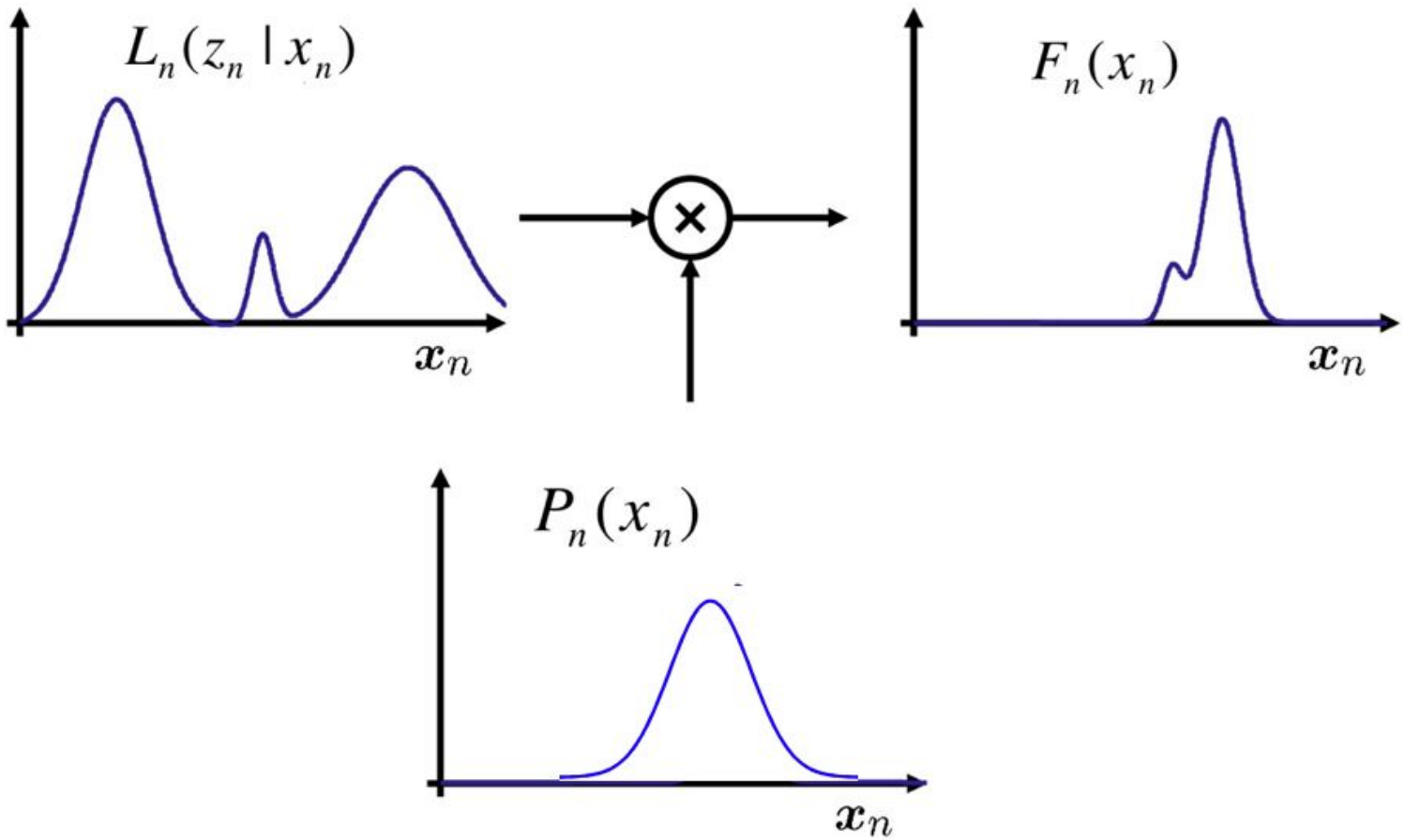
Block Diagram



Block diagram of dynamic filtering.

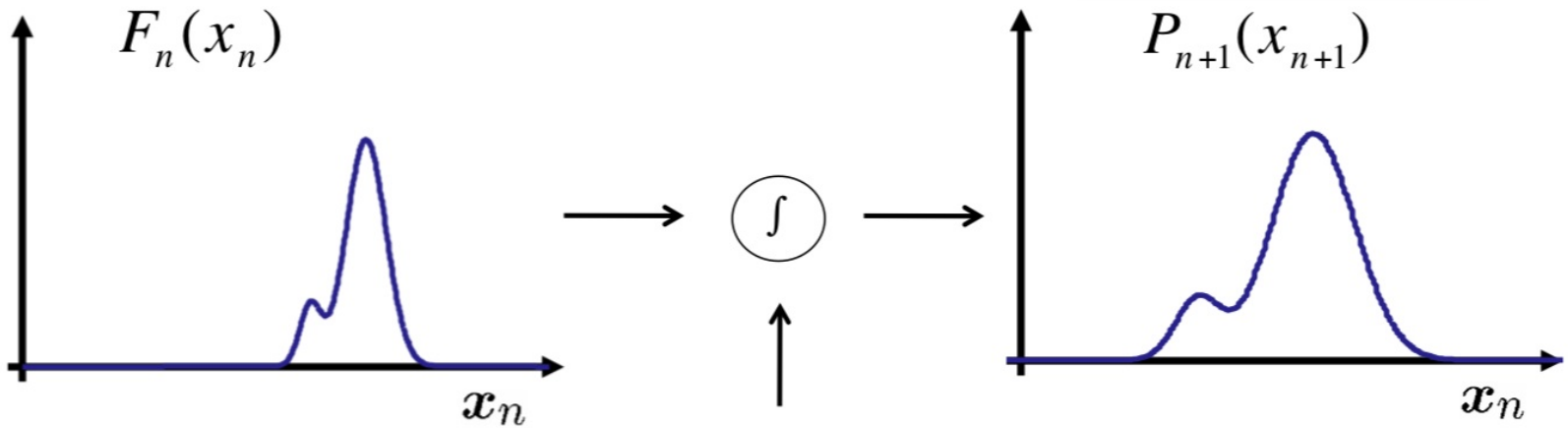
Filtering

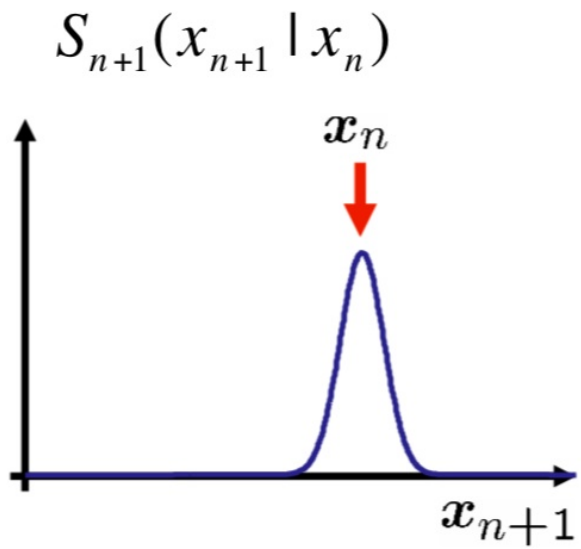
$$F_n(x_n) \propto L_n(z_n|x_n)P_n(x_n)$$



Prediction

$$P_{n+1}(x_{n+1}) = \int S_n(x_{n+1}|x_n) F_n(x_n) \mathrm{d}x_n$$





Estimating x_n

We have many possibilities. Given

$$F_n(x_n) = p(x_n|z_1, \ldots, z_n)$$

we can minimize various risk functions based on a loss and the posterior distribution F_n.

ℓ_2:

$$\begin{aligned}\widehat{x}_n &= \arg\min_{\widetilde{x}} \mathbb{E}_{F_n}[(x_n - \widetilde{x})^2] \\ &= \int x_n F_n(x_n) \mathrm{d}x_n\end{aligned}$$

ℓ_1:

$$\widehat{x}_n = \arg\min_{\widetilde{x}} \mathbb{E}_{F_n}[|x_n - \widetilde{x}|]$$

$\ell_{0/1}$:

$$\widehat{x}_n = \arg\max_{x} F_n(x_n)$$

Gauss-Markov Model

Definition: Gauss-Markov Model

A random process of the form

$$x_n = Ax_{n-1} + Bu_n$$

where $A \in \mathbb{R}^{p\times p}$ and $B \in \mathbb{R}^{p\times q}$, u_n is a sequence of uncorrelated, independent jointly Gaussian vectors with $\mathbb{E}[u_n] = 0$, and the initial state $x_0 \sim \mathcal{N}(\mu_x, C_x)$.

A simple model for x_n which allows us to specify the correlation between samples is the **first-order Gauss-Markov** process model:

$$x_n = ax_{n-1} + u_n, \quad n = 1, 2, \cdots$$

$u_n \sim \mathcal{N}(0, \sigma_u^2)$ (White Gaussian noise process)

To initialize the process we take x_0 to be the realization of a Gaussian random variable:

$$x_0 \sim \mathcal{N}(0, \sigma_x^2)$$

u_n is called the **driving** or **excitation** noise. The model

$$x_n = ax_{n-1} + u_n$$

is called the **dynamical** or **state** model. The current output x_n depends only on the **state** of the system at the previous time, or x_{n-1}, and the current input u_n.

$$\begin{aligned}
x_1 &= ax_0 + u_0 \\
x_2 &= ax_1 + u_1 = a(ax_0 + u_0) + u_1 \\
&= a^2 x_0 + au_1 + u_2 \\
&\vdots \\
x_n &= a^{n+1} x_0 + \sum_{k=1}^{n} a^k u_{n-k}
\end{aligned}$$

What can we say about the statistics of x_n?

First note

$$\mathbb{E}\left[x_n\right] = a^{n+1}\mathbb{E}\left[x_0\right] + \sum_{k=1}^{n} a^k \mathbb{E}\left[u_{n-k}\right] =$$

The correlation can be computed by noting

$$\begin{aligned}
\mathbb{E}\left[x_m x_n\right] &= \mathbb{E}\left[\left(a^{m+1}x_0 + \sum_{k=1}^{m} a^k u_{m-k}\right) \times \left(a^{n+1}x_0 + \sum_{l=1}^{n} a^l u_{m-l}\right)\right] \\
\mathbb{E}\left[x_m x_n\right] &= \mathbb{E}\left[a^{m+n+2}x_0^2\right] + \mathbb{E}\left[\sum_{k=1}^{m}\sum_{l=1}^{n} a^{k+l} u_{m-k}u_{n-l}\right]
\end{aligned}$$

$$\mathbb{E}\left[u_{m-k}u_{n-l}\right] =$$

If $m > n$, then

$$\mathbb{E}\left[x_m x_n\right] = a^{m+n+2}\sigma_x^2 + a^{m-n}\sigma_u^2 \sum_{k=1}^{n} a^{2k}$$

If $|a| > 1$, then it's obvious that the process diverges (variance $\longrightarrow \infty$). So, let's assume $|a| < 1$ and hence a stable system. Thus as m and n get large

$$a^{m+n+2}\sigma_x^2 \to 0$$

Now let $m - n = \tau$. Then for m and n large we have

$$\mathbb{E}\left[x_m x_n\right] = a^{\tau}\sigma_u^2 \underbrace{\sum_{k=1}^{n} a^{2k}}_{\frac{a^2}{1-a^2}} = \frac{a^{\tau+2}\sigma_u^2}{1-a^2}$$

This shows us how correlated the process is.

$$|a| \to 1 \implies \text{heavily correlated (or anticorrelated)}$$
$$|a| \to 0 \implies \text{weakly correlated}$$

Vector case

Let's look at a more general formulation of the problem at hand. Suppose that we have a vector-valued dynamical equation

$$\boldsymbol{x}_{n+1} = A\boldsymbol{x}_n + Bu_n$$

where

$$\begin{aligned}
\boldsymbol{x}_n &\text{ is } && p \times 1 \\
\boldsymbol{x}_0 &\sim && \mathcal{N}(\boldsymbol{\mu}_0, C_0) \\
A &\text{ is } && p \times p \\
B &\text{ is } && p \times q \\
u_n &\text{ is } && q \times 1 \\
u_n &\sim && \mathcal{N}(0, \sigma_u^2) \text{ iid (white Gaussian } \textbf{excitation}\text{)}
\end{aligned}$$

This reduces to the case we just looked at when $p = q = 1$.

p-th order Gauss-Markov processes

$$x_{n+1} = a_1 x_n + a_2 x_{n-1} + \cdots + a_p x_{n-p+1} + u_n$$

Define

$$\boldsymbol{x}_n = \begin{bmatrix} x_{n-p+1} \\ x_{n-p+2} \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Then

$$\boldsymbol{x}_{n+1} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & a_{p-1} & a_p \end{bmatrix}}_{A = \textbf{state transition matrix}} \underbrace{\begin{bmatrix} x_{n-p+1} \\ x_{n-p+2} \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}}_{\boldsymbol{x}_n} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B} u_n$$

Since $\boldsymbol{x}_n$ is a linear combination of Gaussian vectors:

$$\boldsymbol{x}_n = A^n \boldsymbol{x}_0 + \sum_{k=1}^{n} A^{k-1} B u_{n-k}$$

we know that $\boldsymbol{x}_n$ is also Gaussian distributed with mean and covariance $C_n = \mathbb{E}[\boldsymbol{x}_n \boldsymbol{x}_n^T]$:

$$\boldsymbol{x}_n \sim \mathcal{N}(\ , C_n)$$

The covariance can be recursively computed from the basic state equation:

$$C_{n+1} =$$