Lecture 11: Spectra of Sampled Signals

Recall digital frequency \( \hat{f} = \frac{f_0}{f_s} \)

Note that if \( f_s > 2f_{\text{max}} \), then \( \hat{f}_{\text{max}} = \frac{f_{\text{max}}}{f_s} < \frac{1}{2} \) cycles/sample

Recall the Fourier series expansion/representation/synthesis:

\[
x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_0 kt}
\]

If \( x(t) \) only has low frequencies

\[
x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_0 kt}
\]

\( \Rightarrow f_{\text{max}} = Nf_0 \)

\( \Rightarrow \) to avoid aliasing, sample @ freq \( f_s > 2Nf_0 \)
Suppose we sample \( x(t) \) at freq \( f_s = 2Nf_0 \) \((T_s = \frac{1}{2f_s})\) \(^2\) 

\[
x[n] = x(nT_s) = \sum_{k=-N}^{N} a_k e^{j2\pi k \frac{f_0}{f_s} n}
\]

\[
= \sum_{k=-N}^{N} a_k e^{j2\pi \frac{k}{2N} n}
\]

\[
\Rightarrow \text{digital frequencies in } x[n] \text{ are}
\]

\[
0, \pm \frac{1}{2N}, \pm \frac{2}{2N}, \pm \frac{3}{2N}, \ldots, \pm \frac{N-1}{2N}, \pm \frac{1}{2}
\]

\[
\frac{1}{2} = f_s = \frac{1}{T_s} = \frac{f_s}{N}
\]

\[
N = 4
\]

\[
\hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \hat{a}_5, \hat{a}_6, \hat{a}_7, \hat{a}_8, \hat{a}_9, \hat{a}_{10}, \hat{a}_{11}, \hat{a}_{12}
\]

\[
f_{cyc/amp} = f_s = \frac{1}{T_s}
\]
Example:
\[ x(t) = 2 \cos(2\pi 100t) - 3 \cos(2\pi 300t) \]

\[ f_s = 600 \text{ samples/sec} \quad \Rightarrow \quad nT_s = n/f_s = n/600 \]

\[ x[n] = 2 \cos(2\pi \frac{100}{600} n) - 3 \cos(2\pi \frac{300}{600} n) \]

\[ \hat{f}_1 = \frac{1}{6}, \quad \hat{f}_2 = \frac{1}{2} \]

\[ x(f) = \frac{3}{2} \]

\[ -3/2 \]

\[ -\frac{1}{2} \to \frac{1}{2} \]

\[ \Rightarrow \text{same shape as original spectrum} \]

\[ f_s = 350 \text{ samples/sec} \]

\[ x[n] = 2 \cos(2\pi \frac{100}{350} n) - 3 \cos(2\pi \frac{300}{350} n) \]

\[ = 2 \cos(2\pi \frac{3}{7} n) - 3 \cos(2\pi \frac{1}{7} n) \]

\[ \hat{f}_1 = \frac{3}{7}, \quad \hat{f}_2 = \frac{1}{7} \]
Ultimately, we want to reconstruct a signal from samples. Our strategy: assume $x(t)$ is low-frequency and find the lowest-frequency sum of sinusoids that matches the samples. Aliasing makes that process inaccurate.